

STRUCTURE OF CHEVALLEY GROUPS OVER RINGS VIA UNIVERSAL LOCALIZATION

ALEXEI STEPANOV

*Dedicated to my teacher and good friend, Nikolai Vavilov,
on his sixtieth birthday*

ABSTRACT. In the current article we study structure of a Chevalley group $G(R)$ over a commutative ring R . We generalize and improve the following results:

- standard, relative, and multi-relative commutator formulas;
- nilpotent structure of [relative] K_1 ;
- bounded word length of commutators.

To this end we enlarge the elementary group, construct a generic element for the extended elementary group, and use localization in the universal ring. The key step is a construction of a generic element for a principle congruence subgroup, corresponding to a principle ideal.

INTRODUCTION

In the current article we study structure of a Chevalley group $G(R)$ over a commutative ring R . We generalize and improve the following results:

- standard, relative, and multi-relative commutator formulas;
- nilpotent structure of [relative] K_1 ;
- bounded word length of commutators.

The standard commutator formulas for $G = GL_n$ was proved by L. Vaserstein [33] and independently by Z. Borevich and N. Vavilov [9]. In 1987 G. Taddei proved normality of the elementary subgroup in a Chevalley group. It was already known at that time how to deduce the standard commutator formulas from this result using the double of a ring along an ideal and splitting principle. The relative commutator formula was proved in different settings and by different methods in [35, 36, 17, 16]. In [36] it was shown that in most cases it follows from the standard commutator formulas by pure group theoretical arguments. However, the proof of multiple relative commutator formula for $G = GL_n$ from [18] used all the power of Bak's localization procedure.

In the current article for all simply connected Chevalley groups we prove that

$$[\tilde{E}(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq EE(R, \mathfrak{a}, \mathfrak{b}), \text{ where} \\ EE(R, \mathfrak{a}, \mathfrak{b}) = E(R, \mathfrak{a}\mathfrak{b})[E(R, \mathfrak{a}), E(R, \mathfrak{b})],$$

\mathfrak{a} and \mathfrak{b} are ideals of R , and \tilde{E} denotes extended elementary subgroup defined in 8.1. The formula is shown to be stronger than all the commutator formulas mentioned above. Note

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that if $\Phi \neq C_l, G_2$ or 2 is invertible in R , then $\text{EE}(R, \mathfrak{a}, \mathfrak{b}) = [E(R, \mathfrak{a}), E(R, \mathfrak{b})]$, and if $\mathfrak{a} + \mathfrak{b} = R$, then $\text{EE}(R, \mathfrak{a}, \mathfrak{b}) = E(R, \mathfrak{a}\mathfrak{b})$.

Let Y be a subset of an abstract group H and let Σ be a generating set of H . *Word length* (or *width*) of Y with respect to Σ is the smallest integer L such that each element of Y can be written as a product of at most L generators. The word length of the linear elementary group $E_n(R)$ with respect to the set of elementary transvections or to the set of all commutators was studied by D. Carter, G. Keller, K. Dennis, L. Vaserstein, W. van der Kallen, O. Tavgen and others. This is related to computing the Kazhdan constant [24]. The word length of $E(R)$ is known to be finite if R is semilocal (by Gauss decomposition) or $R = \mathbb{Z}$ (by Carter–Keller [10]), but it is infinite for $R = \mathbb{C}[x]$ (van der Kallen [20]). It is amazing that the answer is unknown already for $R = F[x]$, where F is a finite field or a number field.

It turns out that finiteness of the word length of the elementary subgroup $E(R)$ of a Chevalley group with respect to elementary root elements is equivalent to finiteness of the word length of $E(R)$ with respect to commutators (provided that $\Phi \neq C_2, G_2$ or R has no residue fields of 2 elements, otherwise elementary group is not necessarily perfect). This follows from joint results of the author with A. Sivatski [25] (for $G = \text{SL}_n$) and with N. Vavilov [30] (for all simply connected Chevalley groups of rank ≥ 2). It is shown in these papers that the word length of the set of all commutators is finite with respect to elementary root elements provided that the maximal spectrum of the ground ring R has finite combinatorial dimension. The length bound obtained in these articles depended on the dimension.

In the current article we prove that the word length of a commutator $[a, b]$, $a \in G(R, \mathfrak{a})$, $b \in \tilde{E}(R, \mathfrak{b})$ with respect to an arbitrary functorial generating set of $\text{EE}(R, \mathfrak{a}, \mathfrak{b})$ is bounded by a constant, depending only on the root system of G .

One of the motivations to study word length of commutators is the following observations by W. van der Kallen: the group $E_n(R)^\infty / E_n(R^\infty)$ is an obstruction for the finiteness of the word length of $E_n(R)$, where infinite power means the direct product of countably many copies of a ring or a group. The bounded word length of commutators implies that this group is central in $K_1(R^\infty)$.

Recall that K_1 -functor of a Chevalley group G is defined by $K_1^G = G(R)/E(R)$. If $G = \text{GL}_n$ and we pass to the limit, we get the definition $K_1(R) = \lim_{n \rightarrow \infty} K_1^{\text{GL}_n}(R)$ of usual stable K_1 . The group $K_1(R)$ is known to be abelian for an arbitrary (even noncommutative) ring R , see [8]. Nonstable group $K_1^G(R)$ can be nonabelian; examples can be found in [21, 4]. But for a finite dimensional Noetherian ring R the group $K_1^G(R)$ is nilpotent provided that G is simply connected. A nilpotent filtration was obtained by A. Bak in [4] for $G = \text{SL}_n$ and by R. Hazrat and N. Vavilov in [14] for all Chevalley groups. Relative version appeared in [6] by the three authors.

We generalize all results on nilpotent structure of K_1^G mentioned above by proving the following commutator formula:

$$[G(R, \mathfrak{a}_0), G(R, \mathfrak{a}_1), \dots, G(R, \mathfrak{a}_m)] \leq \text{EE}(R, \mathfrak{a}_0 \dots \mathfrak{a}_{m-1}, \mathfrak{a}_m),$$

where Bass-Serre dimension of R is not too big with respect to m .

All the results of the current article are proved uniformly, without computations with individual elements of Chevalley groups. The proof is based on a new version of localization method called universal localization. The idea is fairly simple:

- define the extended elementary subgroup;
- construct a generic element for this subgroup;

- use localization procedure in a versal¹ ring;
- project a result to an arbitrary ring.

Since localization procedure is used only in a versal ring, we call the method “universal localization”.

Our proofs are much easier than previous ones. One of the main concern of the article is to show that one can avoid computations with two denominators called by N. Vavilov the “yoga of commutators”. Of course, to get real bounds for the width of commutators one needs some computations, but it seems that bounds obtained via “yoga of commutators” developed in [30, 12, 16, 13] are far from being optimal anyway. From computational point of view the article is a direct continuation of [29]; we prove all results (except Gauss decomposition) which are not proved in [29] beginning with the standard commutator formulas. Thus, this branch of structure theory of Chevalley groups over rings (Suslin’s local-global principle, normality of elementary subgroup, nilpotent structure of K_1 , bounded width of commutators, and the like) requires only computation of the span $\langle U_P(\mathfrak{a}), U_P^-(\mathfrak{b}) \rangle$ of the unipotent radicals of two opposite parabolic subgroups over ideals $\mathfrak{a}, \mathfrak{b}$. Note that all these results are independent of the root system $\Phi \neq A_1$ and invertibility of structure constants.

Another branch is the normal structure, which requires elementary subgroup to be perfect as well as computation of levels. Both results depend on the root system and invertibility of structure constants. A key computation for this branch was made by M. Stein [28] in 1971. In the next article we plan to prove the normal structure of Chevalley groups using only Stein’s computation and the standard commutator formulas.

1. NOTATION

Groups. The identity element of a group will be denoted by 1. However, if G is a group scheme and R is a ring, then we denote by e_R the identity element of the group $G(R)$. Let a, b, c be elements of a group H . By $a^b = b^{-1}ab$ we denote the element conjugate to a by b . The commutator $aba^{-1}b^{-1}$ is denoted by $[a, b]$. Multi-commutators are left normed, i. e. $[a_1, \dots, a_m] = [[a_1, \dots, a_{m-1}], a_m]$, where $a_1, \dots, a_m \in H$. The following commutator identity will be used in a sequel.

Lemma 1.1. $[a, \prod_{k=1}^l b_k] = \prod_{k=1}^l [a, b_k]^{c_k}$, where $c_k = \left(\prod_{i=1}^{k-1} b_i \right)^{-1}$.

Let A and B be subsets of H . By A^B we denote the normal closure of A by B , i. e. the smallest subgroup of H , containing A and normalized by B . The mixed commutator subgroup $[A, B]$ is a subgroup of H , generated by all commutators $[a, b]$, $a \in A$ and $b \in B$. If $A = \{a\}$ is a singleton, then we drop braces and write $[a, B]$ instead of $[\{a\}, B]$. Mixed multi-commutator subgroups are left-normed.

Rings. Throughout the article the term “ring” stands for a commutative ring with a unit and all ring homomorphisms preserve the units.

Let S be a multiplicative (i. e. multiplicatively closed) subset of a ring R . By $S^{-1}R$ we denote the localization of R with respect to S . The localization homomorphism $R \rightarrow S^{-1}R$ is denoted by λ_S . If $S = \{s^k \mid k \in \mathbb{N}\}$, then we write $\lambda_s : R \rightarrow R_s$ for the localization homomorphism.

A sequence (s_1, \dots, s_m) of elements of R is called unimodular if it generates the unit ideal. The set of all unimodular sequences of length m over R is denoted by $\text{Um}_m(R)$.

¹Unlike “universal”, the term “versal” does not require uniqueness.

An ideal \mathfrak{a} of a ring C is called a splitting ideal if $C = R \oplus \mathfrak{a}$ as additive groups, where R is a subring of C . Of course, in this case $R \cong C/\mathfrak{a}$. Equivalently, \mathfrak{a} is a splitting ideal iff it is a kernel of a retraction $C \rightarrow R \subseteq C$. For example, if $C = R[t]$ is a polynomial ring, then tC is a splitting ideal. Another example of a splitting ideal is the fundamental ideal I of the affine algebra A of a group scheme G . Note that if \mathfrak{a} is a splitting ideal of a ring $C = R \oplus \mathfrak{a}$, then $\mathfrak{a} \otimes_R C'$ is a splitting ideal of $C \otimes_R C'$ for any R -algebra C' .

Let K be a ring and let $\varphi : R \rightarrow R''$ and $\varphi' : R' \rightarrow R''$ be K -algebra homomorphisms. Denote by $\iota : R \rightarrow R \otimes_K R'$ and $\iota' : R' \rightarrow R \otimes_K R'$ the canonical homomorphisms. By the universal property of tensor product there exists a unique K -algebra homomorphism $\varphi'' : R \otimes_K R' \rightarrow R''$ such that $\varphi = \varphi'' \circ \iota$ and $\varphi' = \varphi'' \circ \iota'$. This homomorphism will be denoted by $\varphi \otimes_K \varphi'$ to distinguish it from the natural homomorphism $\varphi \otimes_K \varphi' : R \otimes_K R' \rightarrow R'' \otimes_K R''$. Clearly, $\varphi \otimes_K \varphi' = \text{mult} \circ (\varphi \otimes_K \varphi')$, where “mult” is the multiplication homomorphism. In case $K = \mathbb{Z}$ we write \otimes and $\overline{\otimes}$ instead of \otimes_K and $\overline{\otimes}_K$, respectively.

2. GROUP SCHEMES

Let K be a ring and G an affine group scheme over K . Denote by $A = K[G]$ the affine algebra of G . By definition of an affine scheme, an element $h \in G(R)$ can be identified with a ring homomorphism $h : A \rightarrow R$. We always perform this identification. Denote by $g \in G(A)$ the generic element of G , i.e. the identity map $A \rightarrow A$. An element $h \in G(R)$ induces a group homomorphism $G(h) : G(A) \rightarrow G(R)$ by the rule $G(h)(a) = h \circ a$. Thus, the image of g under $G(h)$ equals h . In the sequel for a ring homomorphism $\varphi : R \rightarrow R'$ we denote by the same symbol φ the induced homomorphism $G(\varphi) : G(R) \rightarrow G(R')$. This can not lead to a confusion as we always can distinguish between two different meanings of φ by the type of its argument. With this convention we have $h(g) = h \circ \text{id}_A = h$.

It is easy to see that the identity element $e_K \in G(\mathbb{Z})$ coincides with the counit map $A \rightarrow K$. Let I denotes the fundamental ideal, i.e. the kernel of e_K . Then, $g \in G(A, I) = \text{Ker } G(e_K)$. Note that e_K is a retraction of the structure map $K \rightarrow A$, hence $A = K \oplus I$ as additive groups. Let \mathfrak{q} be an ideal of a ring R and $\rho_{\mathfrak{q}} : R \rightarrow R/\mathfrak{q}$ the reduction homomorphism. The kernel of the induced map $\rho_{\mathfrak{q}} : G(R) \rightarrow G(R/\mathfrak{q})$ is called the principal congruence subgroup and is denoted by $G(R, \mathfrak{q})$. It is easy to see that $h \in G(R, \mathfrak{q})$ iff the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{h} & R \\ e_K \downarrow & & \downarrow \rho_{\mathfrak{q}} \\ K & \longrightarrow & R/\mathfrak{q} \end{array}$$

And the latter is obviously equivalent to saying that $h(I) \subseteq \mathfrak{q}$. In particular, $\mathfrak{q} = h(I)$ is the smallest ideal such that $h \in G(R, \mathfrak{q})$.

Lemma 2.1. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R , Then $[G(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq G(R, \mathfrak{ab})$.*

Proof. A proof for $G = \text{GL}_n$ can be found in [35] and the general case can be easily deduced from this. However, we prefer to give a direct proof using the formalism of the current subsection.

Note that $A \otimes_K A = (K \oplus I) \otimes_K (K \oplus I) = K \oplus I \otimes_K K \oplus K \otimes_K I \oplus I \otimes_K I$. Therefore, $(A \otimes_K I) \cap (I \otimes_K A) = I \otimes_K I$. There are two natural inclusions of the ring A into $A \otimes_K A$. Denote by g_1 and g_2 the images of the generic element $g \in G(A)$ in $G(A \otimes_K A)$ under these

inclusions. Since, $g_1 \in G(A \otimes_K A, I \otimes_K A)$ and $g_2 \in G(A \otimes_K A, A \otimes_K I)$, we have

$$[g_1, g_2] \in G(A \otimes_K A, I \otimes_K A) \cap G(A \otimes_K A, A \otimes_K I) = \\ G(A \otimes_K A, (I \otimes_K A) \cap (A \otimes_K I)) = G(A \otimes_K A, I \otimes_K I).$$

Now, let $a \in G(R, \mathfrak{a})$ and $b \in G(R, \mathfrak{b})$. Then the map $a \otimes b : A \otimes_K A \rightarrow R$ takes $I \otimes_K A$ into \mathfrak{a} and $A \otimes_K I$ into \mathfrak{b} . The induced group homomorphism $a \otimes b : G(A \otimes_K A) \rightarrow G(R)$ maps g_1 to a and g_2 to b . Therefore,

$$[a, b] = (a \otimes b)([g_1, g_2]) \in (a \otimes b)(G(A \otimes_K A, I \otimes_K I)) \leq G(R, \mathfrak{ab}).$$

□

3. CHEVALLEY GROUPS

From now on G denotes a Chevalley–Demazure group scheme over \mathbb{Z} with a root system $\Phi \neq A_1$. Throughout the article we keep the notation of the previous subsection: $A = \mathbb{Z}[G]$, $g \in G(A)$ is the generic element, and I is the fundamental ideal of A . Denote by E the elementary subgroup (subfunctor) of G . If G is simply connected, then it follows from the Gauss decomposition that over a semilocal ring R the group $E(R)$ coincides with $G(R)$, see [27] or [26] for a more general result. From the very beginning we choose a split maximal torus T ; all root subgroups are assumed to be T -invariant.

Let \mathfrak{q} be an ideal of a ring R . By $E(\mathfrak{q})$ we denote the subgroup of $E(R)$, generated by $x_\alpha(r)$ for all $\alpha \in \Phi$ and $r \in \mathfrak{q}$. The relative elementary subgroup $E(R, \mathfrak{q})$ is the normal closure of $E(\mathfrak{q})$ in $E(R)$.

The following result relates two groups defined above. It was first stated (without a proof) by J. Tits in [32]; a proof appeared in the paper [34] by L. Vaserstein. A more detailed proof using the same idea can be found in [29]. All computations with individual elements of a Chevalley group are hidden in this proof. Dilation principle and normality of elementary subgroup follows easily from this result. We state only a particular case $\mathfrak{q} = tR$ which will be used in the sequel.

Lemma 3.1. *Let $t \in R$. Then $E(R, t^3 R) \leq E(tR)$.*

Another important prerequisite is splitting principle. It seems that idea to use splitting belongs to A. Suslin. However, in [31] he used another consequence of splitting than what we use in the current article. The splitting principle stated below was published in [1] by E. Abe, in [23] by V. Petrov and A. Stavrova (for non-split groups), and in [2] by H. Apte, P. Chattopadhyay, and R. Rao, (relative case). The following statement is the relative version of the splitting principle from [3].

Lemma 3.2. *Let R be a subring of C and \mathfrak{a} an ideal of C such that $C = R \oplus \mathfrak{a}$. Let \mathfrak{b}' be an ideal of R and $\mathfrak{b} = \mathfrak{b}'C$. Then*

$$E(\Phi, C, \mathfrak{b}) \cap G(\Phi, C, \mathfrak{a}) = E(C, \mathfrak{ab})[E(C, \mathfrak{a}), E(C, \mathfrak{b})].$$

In particular, If $\mathfrak{b} = C$, then $E(\Phi, C) \cap G(\Phi, C, \mathfrak{a}) = E(C, \mathfrak{a})$.

For two arbitrary ideals \mathfrak{a} and \mathfrak{b} of a ring C the group in the right hand side of the displayed formula above we introduce special notation:

$$EE(C, \mathfrak{a}, \mathfrak{b}) = E(C, \mathfrak{ab})[E(C, \mathfrak{a}), E(C, \mathfrak{b})].$$

If $\Phi \neq C_l, G_2$ or 2 is invertible in C , then $E(C, \mathfrak{ab}) \leq [E(C, \mathfrak{a}), E(C, \mathfrak{b})]$ (see [29, subsection 1.2]), hence $EE(C, \mathfrak{a}, \mathfrak{b}) = [E(C, \mathfrak{a}), E(C, \mathfrak{b})]$.

4. GAUSS DECOMPOSITION

Fix a split maximal torus T in G and a Borel subgroup, containing this torus. Let U and U^- denote the unipotent radical of this Borel subgroup and its opposite. Denote by W the Weyl group.

Lemma 4.1. *There exist a unimodular set of elements $\{s_w \in A \mid w \in W\}$ such that $\lambda_{s_w}(g) \in wT(A_{s_w})U(A_{s_w})U^-(A_{s_w})$ for all $w \in W$.*

Proof. By [19, II, 1.9] the big cell TUU^- is a principal open subscheme and the Gauss cells $\mathfrak{G}_w = wTUU^-$ form an open cover of G as w ranges over W . Since a Gauss cell is a shift of the big cell, all of them are principal open subschemes, i. e. \mathfrak{G}_w is an affine scheme with affine algebra $\mathbb{Z}[\mathfrak{G}_w] = A_{s_w}$ for some $s_w \in A$. By the definition of an open cover [19, I, 1.7(5)] the set $\{s_w \mid w \in W\}$ is unimodular.

The image $\lambda_{s_w}(g)$ of g in $G(A_{s_w})$ is the localization homomorphism λ_{s_w} itself. This is a tautology that it factors through λ_{s_w} and this means that $\lambda_{s_w}(g) \in \mathfrak{G}_w(A_{s_w})$, i. e. we have the required factorization for $\lambda_{s_w}(g)$. \square

Of course, being an open cover does not mean that the union of the sets of points of all Gauss cells coincides with the group of points of the scheme G . If R is a ring, then an element $a \in G(R)$ lies in $\mathfrak{G}_w(R)$ iff $a(s_w)$ is invertible in R . Thus, we have neither existence nor uniqueness of a Gauss cell, containing a given element. On the other hand, decomposition of $a \in \mathfrak{G}_w(R)$ as a product $a = whuv$, where $h \in T(R)$, $u \in U(R)$, and $v \in U^-(R)$, is unique. This implies the following useful property of the big cell \mathfrak{G}_1 .

Lemma 4.2. *Let \mathfrak{q} be an ideal of a ring R . Then $\mathfrak{G}_1(R) \cap G(R, \mathfrak{q}) = T(R, \mathfrak{q})U(R, \mathfrak{q})U^-(R, \mathfrak{q})$.*

Proof. Let $a = huv \in G(R, \mathfrak{q})$, where $h \in T(R)$, $u \in U(R)$, and $v \in U^-(R)$. Then $\rho_{\mathfrak{q}}(huv) = e$, hence $\rho_{\mathfrak{q}}(hu) = \rho_{\mathfrak{q}}(v)^{-1}$. Since $TU \cap U^-$ and $T \cap U$ both are trivial, we have $\rho_{\mathfrak{q}}(h) = \rho_{\mathfrak{q}}(u) = \rho_{\mathfrak{q}}(v)^{-1} = e$, i. e. $h \in T(R, \mathfrak{q})$, $u \in U(R, \mathfrak{q})$, and $v \in U^-(R, \mathfrak{q})$ as required. \square

If G is simply connected, then its torus is generated by the images of tori of the fundamental SL_2 . Since $T_{\mathrm{SL}_2}(R, \mathfrak{q}) \leq E_2(R, \mathfrak{q})$ (this is the simplest case of the Whitehead lemma), we have $T(R, \mathfrak{q}) \leq E(R, \mathfrak{q})$ for an arbitrary simply connected Chevalley–Demazure group scheme G .

Corollary 4.3. *Suppose that an ideal \mathfrak{q} of a ring R is contained in the Jacobson radical of R . If G is simply connected, then $G(R, \mathfrak{q}) \leq E(R, \mathfrak{q})$.*

Proof. Denote by J the Jacobson radical of R . If $a \in G(R, J)$, then $\rho_J(a) = e \in \mathfrak{G}_1(R/J)$. It follows that $\rho_J(a)(s_1) = \rho_J(a(s_1))$ is invertible in R/J , hence $a(s_1)$ is invertible in R/J which implies that $a \in G_1(R)$. By the previous lemma $a \in T(R, \mathfrak{q})U(R, \mathfrak{q})U^-(R, \mathfrak{q})$. Since G is simply connected, we have $a \in E(R, \mathfrak{q})$. \square

Corollary 4.4. *If G is simply connected, then there exists a unimodular set of elements $\{s_1, \dots, s_l\}$ such that $\lambda_{s_k}(g) \in E(A_{s_k}, I_{s_k})$ for all $k = 1, \dots, l$.*

Proof. Take the unimodular set from the previous lemma and $l = \#W$. Clearly, there exists a representative of w in $E(A_{s_w})$ (in fact even in the image of $E(\mathbb{Z})$). Since G is simply connected, $T \leq E$. Therefore all Gauss cells lie inside the elementary group.

Fix $w \in W$ and let $r = s_w$. Then $\rho_{I_r}(\lambda_r(g)) = \lambda_r(\rho_I(g)) = e \in \mathfrak{G}_w(A_r/I_r)$. But the identity element e_R lies in a Gauss cell $\mathfrak{G}_w(R)$ iff $w = 1$ or R is the trivial ring. It follows that $I_r = A_r$ iff $w \neq 1$, i. e. $s_w \in I$ for all $w \neq 1$. In particular, $\lambda_{s_w}(g) \in E(A_{s_w}, I_{s_w})$ for all $w \neq 1$. For $w = 1$ the result follows from Lemma 4.2. \square

Normality of the elementary subgroup follows easily from Corollary 4.4 and Lemma 3.1, see [29].

5. STANDARD COMMUTATOR FORMULAS

For the results proved in the current article we do not need elementary subgroup $E(R)$ to be perfect, therefore we state standard commutator formulas only as inclusions. These inclusions turn into equalities if $\Phi \neq C_2, G_2$ or R has no residue fields of two elements. Simultaneously with the commutator formulas we prove that certain set of commutators has finite word length with respect to an arbitrary functorial generating set, although this result will be a special case of Theorem 9.1. This is done to illustrate advantages of working with “generic example” at the very beginning.

In this subsection G is Chevalley–Demazure group scheme, not necessarily simply connected.

Proposition 5.1. *Let \mathfrak{q} be an ideal of a ring R . Then*

$$[E(R, \mathfrak{q}), G(R)] \leq E(R, \mathfrak{q}) \quad \text{and} \quad [E(R), G(R, \mathfrak{q})] \leq E(R, \mathfrak{q}).$$

Proof. Let t be an independent variable. For $\alpha \in \Phi$ consider the element $h = [x_\alpha(t), g] \in G(A[t])$. Since the elementary subgroup is normal, we have $h \in E(A[t])$. On the other hand, h vanishes modulo t and modulo $I[t]$. Since both $tA[t]$ and $I[t]$ are splitting ideals, it follows from the splitting principle 3.2 that $h \in E(A[t]) \cap G(A[t], tA[t]) = E(A[t], tA[t])$ and $h \in E(A[t]) \cap G(A[t], I[t]) = E(A[t], I[t])$.

Now, let $a = [x_\alpha(r), b]$ for some $r \in R$ and $b \in G(R)$. Let $\varphi : A[t] \rightarrow R$ be the (unique) extension of $b : A \rightarrow R$, sending t to r . We have $\varphi(g) = b(g) = b$, hence $\varphi(h) = a$. If $r \in \mathfrak{q}$, then $a = \varphi(h) \in \varphi(E(A[t], tA[t])) = E(R, \mathfrak{q})$. Similarly, if $b \in G(R, \mathfrak{q})$, then $b(I) \subseteq \mathfrak{q}$ and $a = \varphi(h) \in \varphi(E(A[t], I[t])) \leq E(R, \mathfrak{q})$. \square

To define the notion of a functorial generating set of $E(R, \mathfrak{q})$ we define the category of ideals. Objects of this category are pairs (R, \mathfrak{q}) , where \mathfrak{q} is an ideal of a ring R . A morphism $\theta : (R, \mathfrak{q}) \rightarrow (R', \mathfrak{q}')$ is a ring homomorphism $R \rightarrow R'$ that maps \mathfrak{q} into \mathfrak{q}' . Let Σ be a functor from the category of ideals to the category of sets. We call it a *functorial generating set* for the relative elementary subgroup if there is a natural inclusion of $\Sigma(R, \mathfrak{q})$ to the underlying set of $E(R, \mathfrak{q})$ and $\Sigma(R, \mathfrak{q})$ spans $E(R, \mathfrak{q})$.

An example of a functorial generating set of the relative elementary subgroup is well known (see [34]). Namely,

$$\Sigma(R, \mathfrak{q}) = \{x_\alpha(p)^{x^{-\alpha(r)}} \mid \alpha \in \Phi, p \in \mathfrak{q}, r \in R\}.$$

Corollary 5.2. *Let Σ be a functorial generating set for the relative elementary subgroup of G . Then there exists $L \in \mathbb{N}$ such that for any ring R and any ideal \mathfrak{q} of R width of the set*

$$\{[x_\alpha(r), b] \mid r \in \mathfrak{q}, b \in G(R) \text{ or } r \in R, b \in G(R, \mathfrak{q})\}$$

with respect to $\Sigma(R, \mathfrak{q})$ is not greater than L .

Proof. Clearly, if $\varphi : (R, \mathfrak{q}) \rightarrow (R', \mathfrak{q}')$ takes $a \in E(R, \mathfrak{q})$ to $b \in E(R', \mathfrak{q}')$, then width of b with respect to $\Sigma(R', \mathfrak{q}')$ is not greater than the width of a with respect to $\Sigma(R, \mathfrak{q})$. Now the result follows from the previous proof as any element of the set under consideration is the image of the element $h \in E(A[t], I[t]) \cap E(A[t], tA[t])$, constructed there. \square

6. KEY CONSTRUCTION

In this section we construct a ring $D = D_G$ together with elements $t \in D$ and $f \in G(D, tD)$ satisfying the following property.

Property 6.1. Given a ring R , an element $r \in R$, and $h \in G(R, rR)$ there exists a homomorphism $\theta : D \rightarrow R$ such that $\theta(t) = r$ and $\theta(f) = h$.

In case of $G = \mathrm{SL}_n$ the construction is very simple. Namely,

$$D_{\mathrm{SL}_n} = \mathbb{Z}[t, y_{ij} \mid 1 \leq i, j \leq n] / (\det(e + ty) - 1),$$

where e denotes the identity matrix and y the matrix with y_{ij} in position (i, j) . The general construction is a kind of a cone over G at the identity element. Denote by $M_n(R)$ the full matrix ring over R , so that M_n defines an affine scheme over \mathbb{Z} isomorphic to the n^2 -dimensional affine space. The affine algebra $\mathbb{Z}[M_n]$ will be identified with the polynomial ring $\mathbb{Z}[z_{ij} \mid 1 \leq i, j \leq n]$. Let $\pi : G \rightarrow M_n$ be a faithful representation of G (which means that π is a closed immersion) and let $\pi^* : \mathbb{Z}[M_n] \rightarrow A$ be the corresponding ring homomorphism. Consider a homomorphism of polynomial rings $\varphi : \mathbb{Z}[M_n] \rightarrow \mathbb{Z}[t, y_{ij} \mid 1 \leq i, j \leq n]$ sending z_{ij} to $\delta_{ij} + ty_{ij}$ for all $1 \leq i, j \leq n$. Define D by the pushout diagram

$$\begin{array}{ccc} \mathbb{Z}[M_n] & \xrightarrow{\varphi} & \mathbb{Z}[t, y_{ij}] \\ \pi^* \downarrow & & \downarrow \xi \\ A & \xrightarrow{f} & D \end{array}$$

i. e. D is the tensor product of $\mathbb{Z}[t, y_{ij}]$ and A over $\mathbb{Z}[M_n]$. By abuse of notation we denote the images of t and y_{ij} 's in D by the same symbols. Define $f \in G(D)$ as the bottom horizontal arrow of the pushout diagram above. The matrix entries of $\pi(f)$ are the images of z_{ij} 's, therefore $\pi(f) = e + ty$, where y is a matrix with entries y_{ij} . In particular, f vanishes modulo t , i. e. $f \in G(D, tD)$.

Proposition 6.2. *The triple (A, t, f) constructed above satisfies property 6.1. Moreover, let \mathfrak{a} be an ideal of a ring R , $r \in R$, and $h \in G(R, r\mathfrak{a})$. Then there exists a homomorphism $\theta : D \rightarrow R$ such that $\theta(t) = r$, $\theta(f) = h$, and $\theta(y_{ij}) \in \mathfrak{a}$ for all $i, j = 1, \dots, n$.*

Proof. By definition, h is a homomorphism $A \rightarrow R$ and $h(\pi^*(z_{ij}))$ are the matrix entries of $\pi(h)$. Since $h \in G(R, r\mathfrak{a})$, we have $h(\pi^*(z_{ij} - \delta_{ij})) = r\tilde{h}_{ij}$ for some $\tilde{h}_{ij} \in \mathfrak{a}$. Define a homomorphism $\psi : \mathbb{Z}[t, y_{ij}] \rightarrow R$ by $\psi(t) = r$ and $\psi(y_{ij}) = \tilde{h}_{ij}$, so that $\psi \circ \varphi = h \circ \pi^*$. By definition of D there exists a unique homomorphism $\theta : D \rightarrow R$ such that $\theta \circ f = h$ and $\theta \circ \xi = \psi$. The former equation shows that $\theta(f) = h$, whereas the latter implies that $\theta(t) = r$ and $\theta(y_{ij}) = \tilde{h}_{ij} \in \mathfrak{a}$. \square

Denote by Y the ideal in D generated by y_{ij} 's. It plays the same role as the fundamental ideal in the affine algebra of the group in the proof of standard commutator formulas. Here we prove that Y is a splitting ideal.

Lemma 6.3. *The ideal Y is a splitting ideal. Let R be a ring and $s \in R$. Denote by Y' the ideal $(Y \otimes R)/(t - s)$ of $(D \otimes R)/(t - s)$. Then Y' is a splitting ideal. Moreover, $(D \otimes R)/(t - s) = R \oplus Y'$ as additive groups.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathbb{Z}[M_n] & \xrightarrow{\varphi} & \mathbb{Z}[t, y_{ij}] \\ \pi^* \downarrow & & \downarrow \rho \\ A & \xrightarrow{e} & \mathbb{Z}[t] \end{array}$$

where e is the identity element of $G(\mathbb{Z}[t])$ and ρ is the reduction homomorphism modulo the ideal, generated by y_{ij} 's. The element $e(\pi^*(z_{ij}))$ is a matrix entry of the identity matrix. Hence it is equal to $\rho_Y(\varphi(z_{ij})) = \rho_Y(\delta_{ij} + ty_{ij}) = \delta_{ij}$. It follows that the diagram commutes,

therefore there exists a unique map $\rho' : D \rightarrow \mathbb{Z}[t]$ such that $\rho' \circ \xi = \rho$ and $\rho' \circ e = f$. The composition

$$\mathbb{Z}[t] \hookrightarrow \mathbb{Z}[t, y_{ij}] \xrightarrow{\xi} D \xrightarrow{\rho'} \mathbb{Z}[t]$$

is identity and the kernel of ρ' equals Y . Thus $D = \mathbb{Z}[t] \oplus Y$. Tensoring the equation with R we get $D \otimes R = R[t] \oplus Y \otimes R$. Factoring out the ideal generated by $t - s$, we have $D \otimes R/(t - s) = R \oplus Y'$. \square

Corollary 6.4. *Let \mathfrak{a} be an ideal of R . With the notation of the previous lemma we have $E(D \otimes R/(t - s), D \otimes \mathfrak{a}/(t - s)) \cap G(D \otimes R/(t - s), Y') = EE(D \otimes R/(t - s), D \otimes \mathfrak{a}/(t - s), Y')$.*

Proof. The corollary follows immediately from the previous lemma and splitting principle 3.2. \square

7. EXTENDED DILATION PRINCIPLE AND KEY LEMMA

In the current section G is assumed to be simply connected. We prove the Key Lemma of the current article which is a stronger version of Bak's key lemma. The original statement obtained by A. Bak in [4] for $G = \mathrm{GL}_n$ is a special case of Key Lemma 7.3 with $R' = \mathfrak{q} = R$ and $\varphi = \mathrm{id}$. Later Bak's key lemma was extended to all Chevalley groups by R. Hazrat and N. Vavilov in [14]. Relative version appeared in [6] by the three authors. In these articles it was proved using double localization – the method that required commutator calculus called “yoga of commutators”. The lemma is a key computational step in existing proofs of nilpotent structure of K_1 and bounded width of commutators.

If we replace $G(R, s^m R)$ by $E(R, s^m R)$, then the usual dilation principle works (see [29]). Namely, it suffices to consider a generator of $E(s^m R)$ and, since $\mathbb{Z}[X_\alpha] = \mathbb{Z}[t]$, we have a generic element $x_\alpha(t) \in E(R[t], tR[t])$ for such a generator. Afterwards we specialize the independent variable t to s^m to get the result.

For the general case we constructed a generic element of $G(R, s^m R)$ in the previous section. But now t is not an independent variable and we can not apply the usual dilation principle. To get around this problem we prove an extension of the principle. We shall see that over localized ring the principle follows almost immediately from Lemma 3.1. The main concern of the proof is to pull the result back to the group over original ring. This is done in the following lemma which uses method by A. Bak and the author of struggling against zero divisors.

Lemma 7.1. *Let \mathfrak{q} be an ideal of a ring R , $s \in R$, and $a \in G(R, s\mathfrak{q})$. If $\lambda_s(a) \in \lambda_s(E(R, s\mathfrak{q}))$, then $a \in E(R, s\mathfrak{q})$.*

Proof. Let N denote the nilpotent radical of R , $\bar{s} = \rho_N(a)$, $\bar{\mathfrak{q}} = \rho_N(\mathfrak{q})$, and $\bar{a} = \rho_N(a) \in G(R/N, \bar{s}\bar{\mathfrak{q}})$. Since R_s/N_s is naturally isomorphic to $(R/N)_{\bar{s}}$, we have $\lambda_{\bar{s}}(\bar{a}) \in \lambda_{\bar{s}}(E(R/N, \bar{s}\bar{\mathfrak{q}}))$. Since R/N is a reduced ring, by [7, Lemma 5.5] the restriction of $\lambda_{\bar{s}}$ on $\bar{s}R/N$ is injective. Therefore $\bar{a} \in E(R/N, \bar{s}\bar{\mathfrak{q}})$. Since E preserves surjective homomorphisms, there exists $b \in E(R, s\mathfrak{q})$ such that $\rho_N(b) = \bar{a}$. Now, $a \in bG(R, N \cap s\mathfrak{q})$ and $G(R, N \cap s\mathfrak{q})$ is contained in $E(R, s\mathfrak{q})$ by Lemma 4.3. \square

Lemma 7.2. *Let R be a ring, $s, t \in R$ and $a \in G(R, tR)$. Suppose that $\lambda_s(a) \in E(R_s, tR_s)$. Then there exists $m \in \mathbb{N}$ such that $\rho(a) \in E(R/(t - s^m))$, where $\rho : R \rightarrow R/(t - s^m)$ is the reduction homomorphism.*

Proof. Let $R' = R[u]/(t - u^3)$, $\iota : R \rightarrow R'$ be the canonical inclusion, and $a' = \iota(a) \in G(R', tR')$. Clearly, $\lambda_s(a') \in E(R'_s, u^3R'_s)$. By Lemma 3.1 $E(R'_s, u^3R'_s) \leq E(uR'_s)$, hence $\lambda_s(a') = \prod_i x_{\alpha_i}(\frac{ur_i}{s^n})$ for some $\alpha_i \in \Phi$ and $r_i \in R$. Put $m = 3(n+1)$. Then the homomorphism

ρ is the composition of ι with the reduction homomorphism modulo $(u - s^{n+1})R'$. Put $s' = \rho(s)$. Since a localization homomorphism commutes with a reduction homomorphism, we have

$$\lambda_{s'}(\rho(a)) = \prod_i x_{\alpha_i}(s' r'_i) \in \lambda_{s'}(E(s'R/(t - s^m))).$$

On the other hand, $\rho(a) \in G(R/(t - s^m), s'R/(t - s^m))$. Thus, by Lemma 7.1 we have $\rho(a) \in E(R/(t - s^m))$. \square

Now we are at position to prove the key technical statement.

Key Lemma 7.3. *Let R be a ring, $a \in G(R)$, and $s \in R$. Suppose that $\lambda_s(a) \in E(R_s)$. Then there exists $m \in \mathbb{N}$ such that for any ring R' , homomorphism $\varphi : R \rightarrow R'$ and ideal \mathfrak{q} of R' we have $[\varphi(a), G(R', \varphi(s)^m \mathfrak{q})] \leq E(R', \mathfrak{q})$.*

Proof. First, we consider the commutator $[a, f] \in G(D \otimes R)$, where D is the ring, constructed in the previous section. Rings R and D will be identified with their canonical images in $D \otimes R$. By the standard commutator formula 5.1 $\lambda_s([a, f]) \in E(D \otimes R_s, tD \otimes R_s)$. By Lemma 7.2 there exists $m \in \mathbb{N}$ such that $\rho([a, f]) \in E(D \otimes R/(t - s^m))$, where $\rho = \rho_{(t-s^m)}$ is the reduction homomorphism. Let Y' be the ideal of $D \otimes R/(t - s^m)$, generated by the images of y_{ij} 's. Since, f vanishes modulo Y , we have $\rho([a, f]) \in G(D \otimes R/(t - s^m), Y')$. By Lemma 6.4 we have $\rho([a, f]) \in E(D \otimes R/(t - s^m), Y')$, hence $[a, f] \in E(D \otimes R, Y)G(D \otimes R, (t - s^m))$.

Now, let $b \in G(R', \varphi(s)^m \mathfrak{q})$. By Proposition 6.2 there exists a homomorphism $\theta : D \rightarrow R'$ such that $\theta(t) = \varphi(s)^m$, $\theta(f) = b$, and $\theta(Y) \subseteq \mathfrak{q}$. The homomorphism $\theta \otimes \varphi : D \otimes R \rightarrow R'$ maps t and s^m to the same element of R' , hence it $\theta \otimes \varphi(G(D \otimes R, (t - s^m))) = \{e\}$. Thus,

$$[\varphi(a), b] = \theta \otimes \varphi([a, f]) \in \theta \otimes \varphi(E(D \otimes R, Y)) \leq E(R', \mathfrak{q}).$$

\square

8. EXTENDED ELEMENTARY GROUP

In this section we construct an extended (relative) elementary group and a generic element for this group. Let s_1, \dots, s_l be a unimodular sequence of elements of the affine algebra A such that $\lambda_{s_k}(g) \in E(A_{s_k})$ for all $k = 1, \dots, l$. Such a sequence exists by Corollary 4.4. Denote by $\text{Um}_l(R)$ the set of all unimodular sequences over R of length l .

Definition 8.1. Let \mathfrak{a} be an ideal of a ring R . Define an extended relative elementary group $\tilde{E}(R, \mathfrak{a})$ by the formula

$$\tilde{E}(R, \mathfrak{a}) = \bigcap_{(r_1, \dots, r_l) \in \text{Um}_l(R)} \left(\prod_{k=1}^l G(R, r_k \mathfrak{a}) \right).$$

Formally the definition of \tilde{E} depends on l . In this article we put $l = \#W$ as in Corollary 4.4.

Lemma 8.2. $E(R, \mathfrak{a}) \leq \tilde{E}(R, \mathfrak{a})$.

Proof. Clearly, $\tilde{E}(R, \mathfrak{a})$ is normal in $G(R)$. Therefore, it suffices to show that it contains $x_\alpha(q)$ for all $\alpha \in \Phi$ and $q \in \mathfrak{a}$. If $\sum_{k=1}^l p_k r_k = 1$, then $x_\alpha(q) = \prod_{k=1}^l x_\alpha(p_k r_k q) \in \prod_{k=1}^l G(R, r_k \mathfrak{a})$, which implies the result. \square

Let $D^{(k)}$ denote the ring isomorphic to D generated by $t^{(k)}$ and $y_{ij}^{(k)}$ instead of t and y_{ij} (where $1 \leq i, j \leq n$). Similarly, we write $f^{(k)} \in G(D^{(k)})$ instead of $f \in G(D)$. Let

$$U = \bigotimes_{k=1}^l D^{(k)}.$$

We identify elements of each $D^{(k)}$ with their canonical images in U . Similarly, group elements of $G(D^{(k)})$ are identified with their images in $G(U)$. Denote by $Y^{(k)}$ the ideal of U generated by $y_{ij}^{(k)}$ for all $1 \leq i, j \leq n$ and let $\tilde{Y} = \sum_{k=1}^l Y^{(k)}$. We show that the element $u = f^{(1)} \cdots f^{(l)} \in G(U)$ is a generic element of \tilde{E} in the following sense.

Lemma 8.3. *Given a ring R , a unimodular sequence $r_1, \dots, r_l \in R$, and $b \in \tilde{E}(R, \mathfrak{q})$, there exists a homomorphism $\eta : U \rightarrow R$ such that $\eta(u) = b$, $\eta(t^{(k)}) = r_k$ for all $k = 1, \dots, l$ and $\eta(\tilde{Y}) \subseteq \mathfrak{q}$.*

Proof. By definition of $\tilde{E}(R, \mathfrak{q})$ we can write b as a product of $b^{(k)} \in G(R, r_k \mathfrak{q})$ as k ranges from 1 up to l . By Property 6.1 for each $k = 1, \dots, l$ there exists a homomorphism $\theta_k : D^{(k)} \rightarrow R$ sending $t^{(k)}$ to r_k and $f^{(k)}$ to $b^{(k)}$. Moreover, by Proposition 6.2 $\theta_k(y_{ij}^{(k)}) \in \mathfrak{q}$. By the universal property of tensor product there exists a unique homomorphism $\eta : U \rightarrow R$ such that each θ_k factors through η . It follows that η satisfies the conditions of the lemma. \square

9. LENGTH OF COMMUTATORS

Second important property of the extended elementary group is the commutator formula from the following theorem. We prove it together with the simplest applications of our constructions for the length of commutators. Let Σ be a functorial generating set for the relative elementary subgroup (it is defined before corollary 5.2).

Theorem 9.1. *Let \mathfrak{q} be an ideal of a ring R . Suppose that G is simply connected. Then*

$$[G(R), \tilde{E}(R, \mathfrak{q})] \leq E(R, \mathfrak{q}).$$

Moreover, there exist a constant $L \in \mathbb{N}$ such that for any ring R , ideal \mathfrak{q} of R , $a \in G(R)$, and $b \in \tilde{E}(R, \mathfrak{q})$ the length of the commutator $[a, b]$ with respect to $\Sigma(R, \mathfrak{q})$ is at most L .

Proof. By Lemma 4.4 there exists a unimodular sequence $s_1, \dots, s_l \in A$ such that $\lambda_{s_k}(g) \in E(A_{s_k})$. For each $k = 1, \dots, l$ take $m_k \in \mathbb{N}$ satisfying conditions of Key Lemma 7.3 with $R = A$, $a = g$, and $s = s_k$. Denote by T the ideal of $U \otimes A$ generated by $t^{(k)} - s_k^{m_k}$ for all $k = 1, \dots, l$ (as usual we identify elements of U and A with their images in $U \otimes A$). Denote by R' the quotient ring $U \otimes A / T$. Let $\varphi : A \rightarrow R'$ and $\psi : U \rightarrow R'$ be the canonical homomorphisms. Note that $\psi \otimes \varphi$ is the reduction homomorphism modulo T , in particular it is surjective. Put $h^{(k)} = \psi(f^{(k)})$ and $Y' = \psi \otimes \varphi(\tilde{Y} \otimes A)$. Since $f^{(k)} \in G(D^{(k)}, t^{(k)} Y^{(k)})$ and $\psi(t^{(k)}) = \varphi(s_k)^{m_k}$, we have $h^{(k)} \in G(R', \varphi(s_k)^{m_k} Y')$. By the choice of m_k we have $[\varphi(g), h^{(k)}] \in E(R', Y')$. Since $E(R', Y')$ is normal in $G(R')$, commutator identity 1.1 implies that $[\varphi(g), \psi(u)] \in E(R', Y')$. Therefore, $[g, u] \in vG(U \otimes A, T)$ for some $v \in E(U \otimes A, \tilde{Y} \otimes A)$. Denote by L the length of v with respect to $\Sigma(U \otimes A, \tilde{Y} \otimes A)$.

Now, we construct a homomorphism $\pi : U \otimes A \rightarrow R$ that maps g to a , u to b , $\tilde{Y} \otimes A$ to \mathfrak{q} , and T to zero. Recall, that a itself is a ring homomorphism $A \rightarrow R$ such that $a(g) = a$. For all $k = 1, \dots, l$ denote $r_k = a(s_k^{m_k})$. Since s_1, \dots, s_l is a unimodular sequence, $s_1^{m_1}, \dots, s_l^{m_l}$ is also unimodular (otherwise they were contained in a maximal ideal, but then s_1, \dots, s_l were contained in the same ideal, a contradiction). Hence, its image r_1, \dots, r_l is unimodular as

well. By Lemma 8.3 there exists a homomorphism $\eta : U \rightarrow R$ such that $\eta(u) = b$, $\eta(t^{(k)}) = r_k$ for all $k = 1, \dots, l$ and $\eta(\tilde{Y}) \subseteq \mathfrak{q}$. Put $\pi = \eta \overline{\otimes} a$. Then $\pi(g) = a(g) = a$, $\pi(u) = \eta(u) = b$, $\pi(\tilde{Y} \otimes A) = \eta(\tilde{Y}) \subseteq \mathfrak{q}$, and $\pi(t^{(k)} - s_k^{m_k}) = \eta(t^{(k)}) - a(s_k^{m_k}) = 0$ as required. Finally,

$$[a, b] = \pi([g, u]) = \pi(v) \in \pi(E(U \otimes A, \tilde{Y} \otimes A)) \leq E(R, \mathfrak{q})$$

and length of $[a, b]$ with respect to $\Sigma(R, \mathfrak{q})$ is bounded by the length of v with respect to $\Sigma(U \otimes A, \tilde{Y} \otimes A)$. \square

The following statement is an immediate corollary of the theorem.

Corollary 9.2. *Let G be a simply connected Chevalley group. There exists an integer L depending only on G satisfying the following property: Given $a \in G(R)$ and $b \in E(R)$ the commutator $[a, b]$ can be written as a product of at most L elementary root unipotent elements.*

10. RELATIVE KEY LEMMA

In this section we prepare technical tools for relative commutator formula 11.1 by proving a relative analog of Lemma 7.3. Let \mathfrak{q} be an ideal of a ring R and $t \in R$. Denote by $E(tR, t\mathfrak{q})$ the normal closure of $E(t\mathfrak{q})$ in $E(tR)$. The following result is obtained in [29].

Lemma 10.1. $EE(R, \mathfrak{q}, t^{27}R) \leq E(t\mathfrak{q}, tR)$.

The following statement is a suitable generalization of the relative dilation principle proved in [2], [3], and [29].

Lemma 10.2. *Let \mathfrak{q} be an ideal of a ring R , $s, t \in R$, and $a \in G(R, tR)$. Suppose that $\lambda_s(a) \in EE(R_s, tR_s, \mathfrak{q})$. Then there exists $m \in \mathbb{N}$ such that $\rho(a) \in E(R/(t - s^m), \rho(\mathfrak{q}))$, where $\rho = \rho_{(t-s^m)} : R \rightarrow R/(t - s^m)$ is the reduction homomorphism.*

Proof. The proof is essentially the same as for Lemma 7.2 using Lemma 10.1 instead of Lemma 3.1. \square

Now we prove the relative commutator formula which is a common generalization of commutator formulas 5.1. By different methods and in different settings it was obtained in [35], [36], and [17]. Since we do not want to exclude cases when the elementary group is not perfect, we give yet another proof based on splitting principle 3.2.

Lemma 10.3. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R . Then*

$$[E(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq EE(R, \mathfrak{a}, \mathfrak{b}).$$

Proof. Consider the ring $R \otimes A$. As before, we identify elements of R , A , $G(R)$, and $G(A)$ with their canonical images in $R \otimes A$ and $G(R \otimes A)$. By the first standard commutator formula of Lemma 5.1 we have $[E(R, \mathfrak{a}), g] \leq E(R \otimes A, \mathfrak{a} \otimes A)$. On the other hand, since $g \in G(A, I)$, we have $[E(R, \mathfrak{a}), g] \leq G(R \otimes A, R \otimes I)$. Since $R \otimes I$ is a splitting ideal of $R \otimes A$, by Lemma 3.2 we have $[E(R, \mathfrak{a}), g] \leq EE(R \otimes A, \mathfrak{a} \otimes A, R \otimes I)$.

Now, let $b \in G(R, \mathfrak{b})$. Applying $\text{id}_R \overline{\otimes} b$ to the inclusion above we obtain $[E(R, \mathfrak{a}), b] \leq EE(R, \mathfrak{a}, \mathfrak{b})$ which completes the proof. \square

The following statement generalizes Key Lemma 7.3.

Lemma 10.4. *Let R be a ring, $a \in G(R, \mathfrak{a})$, and $s \in R$. Suppose that $\lambda_s(a) \in E(R_s, \mathfrak{a}_s)$. Then there exists $m \in \mathbb{N}$ such that for any ring R' , homomorphism $\varphi : R \rightarrow R'$, and ideals $\mathfrak{a}' \supseteq \varphi(\mathfrak{a})$ and \mathfrak{b}' of R' we have $[\varphi(a), G(R', \varphi(s)^m \mathfrak{b}')] \leq EE(R', \mathfrak{a}', \mathfrak{b}')$.*

Proof. First, we consider the commutator $[a, f] \in G(D \otimes R)$, where D is the ring, constructed in section 6. Rings R and D will be identified with their canonical images in $D \otimes R$. Since $f \in tD \otimes R$, by the relative commutator formula 10.3 we have $\lambda_s([a, f]) \in \text{EE}(D \otimes R_s, tD \otimes R_s, D \otimes \mathfrak{a})$. By Lemma 10.2 there exists $m \in \mathbb{N}$ such that $\rho([a, f]) \in E(D \otimes R/(t - s^m), \rho(D \otimes \mathfrak{a}))$, where $\rho = \rho_{(t-s^m)}$ is the reduction homomorphism. Let Y' be the ideal of $D \otimes R/(t - s^m)$, generated by the images of y_{ij} 's. Since, f vanishes modulo Y , we have $\rho([a, f]) \in G(D \otimes R/(t - s^m), Y')$. By Lemma 6.4 we have $\rho([a, f]) \in \text{EE}(D \otimes R/(t - s^m), \rho(D \otimes \mathfrak{a}), Y')$, hence $[a, f] \in \text{EE}(D \otimes R, D \otimes \mathfrak{a}, Y)G(D \otimes R, (t - s^m))$.

Now, let $b \in G(R', \varphi(s)^m \mathfrak{b}')$. By Proposition 6.2 there exists a homomorphism $\theta : D \rightarrow R'$ such that $\theta(t) = \varphi(s)^m$, $\theta(f) = b$, and $\theta(Y) \subseteq \mathfrak{b}'$. The homomorphism $\theta \otimes \varphi : D \otimes R \rightarrow R'$ maps t and s^m to the same element of R' , hence $\theta \otimes \varphi(G(D \otimes R, (t - s^m))) = \{e\}$. Thus,

$$[\varphi(a), b] = \theta \otimes \varphi([a, f]) \in \theta \otimes \varphi(\text{EE}(D \otimes R, D \otimes \mathfrak{a}, Y)) \leq \text{EE}(R', \mathfrak{a}', \mathfrak{b}').$$

□

11. RELATIVE COMMUTATOR FORMULAS

In this section for a simply connected Chevalley group we prove a stronger version of relative commutator formula 10.3 together with a bound for the length of commutators. As a consequence we prove the multi-commutator formula from [18].

First, we define a functorial generating set for the group functor EE. Clearly, EE is a functor on the category of pairs of ideals which is defined as follows. An object is a triple $(R, \mathfrak{a}, \mathfrak{b})$, where \mathfrak{a} and \mathfrak{b} are ideals of a ring R . A morphism $\varphi : (R, \mathfrak{a}, \mathfrak{b}) \rightarrow (R', \mathfrak{a}', \mathfrak{b}')$ in this category is a ring homomorphism $\varphi : R \rightarrow R'$ such that $\varphi(\mathfrak{a}) \subseteq \mathfrak{a}'$ and $\varphi(\mathfrak{b}) \subseteq \mathfrak{b}'$. A functor Σ from the category of pairs of ideals to the category of sets is called a functorial generating set for EE if $\Sigma(R, \mathfrak{a}, \mathfrak{b})$ is a generating subset of $\text{EE}(R, \mathfrak{a}, \mathfrak{b})$ for any object $(R, \mathfrak{a}, \mathfrak{b})$ of the category. A natural functorial generating set for EE is obtained in [15]

Theorem 11.1. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R . Suppose that G is simply connected. Then*

$$[\tilde{E}(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq \text{EE}(R, \mathfrak{a}, \mathfrak{b}).$$

Moreover, there exist a constant $L \in \mathbb{N}$ such that for any ring R , ideals \mathfrak{a} and \mathfrak{b} of R , $a \in G(R, \mathfrak{b})$, and $b \in \tilde{E}(R, \mathfrak{a})$ the length of the commutator $[a, b]$ with respect to $\Sigma(R, \mathfrak{a}, \mathfrak{b})$ is at most L .

Proof. By Lemma 4.4 there exists a unimodular sequence $s_1, \dots, s_l \in A$ such that $\lambda_{s_k}(g) \in E(A_{s_k}, I_{s_k})$. For each $k = 1, \dots, l$ take $m_k \in \mathbb{N}$ satisfying conditions of Key Lemma 10.4 with $R = A$, $a = g$, and $s = s_k$. Denote by T the ideal of $U \otimes A$ generated by $t^{(k)} - s_k^{m_k}$ for all $k = 1, \dots, l$ (as usually we identify elements of U and A with their images in $U \otimes A$). Denote by R' the quotient ring $U \otimes A/T$. Let $\varphi : A \rightarrow R'$ and $\psi : U \rightarrow R'$ be the canonical homomorphisms. Note that $\psi \otimes \varphi$ is the reduction homomorphism modulo T , in particular it is surjective. Put $h^{(k)} = \psi(f^{(k)})$ and $Y' = \psi \otimes \varphi(\tilde{Y} \otimes A)$ and $I' = \psi \otimes \varphi(U \otimes I)$. Since $f^{(k)} \in G(D^{(k)}, t^{(k)}Y^{(k)})$ and $\psi(t^{(k)}) = \varphi(s_k)^{m_k}$, we have $h^{(k)} \in G(R', \varphi(s_k)^{m_k}Y')$. By the choice of m_k we have $[\varphi(g), h^{(k)}] \in \text{EE}(R', Y', I')$. Since $\text{EE}(R', Y', I')$ is normal in $G(R')$, commutator identity 1.1 implies that $[\varphi(g), \psi(u)] \in E(R', Y')$. Therefore, $[g, u] \in vG(U \otimes A, T)$ for some $v \in \text{EE}(U \otimes A, \tilde{Y} \otimes A, U \otimes I)$. Denote by L the length of v with respect to $\Sigma(U \otimes A, \tilde{Y} \otimes A, U \otimes I)$.

Now, we construct a homomorphism $\pi : U \otimes A \rightarrow R$ that maps g to a , u to b , $\tilde{Y} \otimes A$ into \mathfrak{a} , $U \otimes I$ into \mathfrak{b} , and T to zero. Recall, that a itself is a ring homomorphism $A \rightarrow R$ such that $a(g) = a$ and $a(I) \subseteq \mathfrak{b}$. For all $k = 1, \dots, l$ denote $r_k = a(s_k^{m_k})$. Since s_1, \dots, s_l

is a unimodular sequence, r_1, \dots, r_l is unimodular as well. By Lemma 8.3 there exists a homomorphism $\eta : U \rightarrow R$ such that $\eta(u) = b$, $\eta(t^{(k)}) = r_k$ for all $k = 1, \dots, l$ and $\eta(\tilde{Y}) \subseteq \mathfrak{q}$. Put $\pi = \eta \otimes a$. Then $\pi(g) = a(g) = a$, $\pi(u) = \eta(u) = b$, $\pi(\tilde{Y} \otimes A) = \eta(\tilde{Y}) \subseteq \mathfrak{a}$, $\pi(U \otimes I) = a(I) \subseteq \mathfrak{b}$ and $\pi(t^{(k)} - s_k^{m_k}) = \eta(t^{(k)}) - a(s_k^{m_k}) = 0$ as required. Finally,

$$[a, b] = \pi([g, u]) = \pi(v) \in \pi(\text{EE}(U \otimes A, \tilde{Y} \otimes A, U \otimes I)) \leq E(R, \mathfrak{a}, \mathfrak{b})$$

and length of $[a, b]$ with respect to $\Sigma(R, \mathfrak{a}, \mathfrak{b})$ is bounded by the length of v with respect to $\Sigma(U \otimes A, \tilde{Y} \otimes A, U \otimes I)$. \square

Next, we observe a handy property of \tilde{E} . Here we do not require G to be simply connected.

Lemma 11.2. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R . Then $[\tilde{E}(R, \mathfrak{a}), G(R, \mathfrak{b})]$ and $\text{EE}(R, \mathfrak{a}, \mathfrak{b})$ are contained in $\tilde{E}(R, \mathfrak{a}\mathfrak{b})$.*

Proof. Let r_1, \dots, r_l be a unimodular sequence. Using Lemma 2.1 we have

$$\left[\prod_{k=1}^l G(R, r_k \mathfrak{a}), G(R, \mathfrak{b}) \right] = \prod_{k=1}^l [G(R, r_k \mathfrak{a}), G(R, \mathfrak{b})] = \prod_{k=1}^l G(R, r_k \mathfrak{a}\mathfrak{b}).$$

This proves that $[\tilde{E}(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq \tilde{E}(R, \mathfrak{a}\mathfrak{b})$. By Lemma 8.2 $E(R, \mathfrak{a}\mathfrak{b}) \leq \tilde{E}(R, \mathfrak{a}\mathfrak{b})$, hence $\text{EE}(R, \mathfrak{a}, \mathfrak{b}) \leq \tilde{E}(R, \mathfrak{a}\mathfrak{b})$. \square

The following multi-commutator formula for $G = \text{GL}_n$ was obtained by R.Hazrat and Z.Zuhong in [18]: If $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ are ideals of a ring R , then

$$[E(R, \mathfrak{a}_1), G(R, \mathfrak{a}_2), \dots, G(R, \mathfrak{a}_m)] = [E(R, \mathfrak{a}_1), \dots, E(R, \mathfrak{a}_m)] = [E(R, \mathfrak{a}_1 \dots \mathfrak{a}_{m-1}), E(R, \mathfrak{a}_m)].$$

We prove even better result for any simply connected Chevalley group G replacing $E(R, \mathfrak{a}_1)$ by $\tilde{E}(R, \mathfrak{a}_1)$ in the left hand side of the formula. However, this stronger formula probably does not hold if a torus in G does not belong to the elementary group.

Theorem 11.3. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_m$ be ideals of a ring R . Then*

$$[\tilde{E}(R, \mathfrak{a}_1), G(R, \mathfrak{a}_2), \dots, G(R, \mathfrak{a}_m)] \leq \text{EE}(R, \mathfrak{a}_1 \dots \mathfrak{a}_{m-1}, \mathfrak{a}_m).$$

Proof. Applying Lemma 11.2 $m - 2$ times we get

$$[\tilde{E}(R, \mathfrak{a}_1), G(R, \mathfrak{a}_2), \dots, G(R, \mathfrak{a}_m)] \leq [\tilde{E}(R, \mathfrak{a}_1 \dots \mathfrak{a}_{m-1}), G(\mathfrak{a}_m)].$$

Now the result follows from Theorem 11.1. \square

It follows from the proof of the previous theorem that length of multi-commutators is bounded by at most the same constant as length of relative commutators in Theorem 11.1.

12. NILPOTENT STRUCTURE OF K_1

In this section G is assumed to be simply connected. We prove a multi-relative version of Bak's theorem on nilpotent structure of K_1 . Actually, we concentrate our attention on the induction step of the proof. The base of induction follows from bi-relative version of surjective stability which is available at the time only for the special linear group by A. Mason and W. Stothers [22]. That is why we use an axiomatic notion of dimension function instead of actual dimension of a ring. The set of axioms is similar to those of dimension function of A. Bak [5, 11] for a particular kind of infrastructure and structure squares.

Definition 12.1. Let $\delta = \delta_G$ be a function from the class of rings to $\mathbb{Z} \cup \{\infty\}$ satisfying the following properties.

1. If $\delta(R) \leq 1$, then for any ideals \mathfrak{a} and \mathfrak{b} of R we have $[G(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq \text{EE}(R, \mathfrak{a}, \mathfrak{b})$.
2. Given an ideal \mathfrak{a} of a ring R there exists a multiplicative subset S of R such that $G(S^{-1}R, S^{-1}\mathfrak{a}) = E(S^{-1}R, S^{-1}\mathfrak{a})$ and $\delta(R/rR) < \delta(R)$ for all $r \in S$.
3. If R is not Noetherian, then $\delta(R) = \infty$.

Then δ is called a dimension function for G .

For example dimension of maximal spectrum $\dim \text{Max } R$ of a ring R is a dimension function in this sense. Indeed, $\dim \text{Max } R = 0$ iff R is semilocal and for a semilocal ring $G(R, \mathfrak{a}) = E(R, \mathfrak{a})$. Therefore, the first property follows from Corollary 12.4. For the second, take $S = R \setminus \bigcup_i \mathfrak{p}_i$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_j$ are all minimal prime ideals of R , see [8, Chapter 3]. A little bit smaller function, Bass-Serre dimension BS-dim , also is a dimension function in our sense, see [4, 11] for details. Clearly, the second and third properties does not depend on a shift of dimension function. Namely, if (2) holds for δ then it holds also for $\delta' = \delta - k$, where k is a given integer. A. Mason and W. Stothers [22] showed that for $G = \text{SL}_n$ the function $\text{BS-dim}(R) - n + 2$ satisfies the first property. It is expected that in general $\text{BS-dim}(R) - \text{rank } G + 1$ also is a dimension function for G , but we can not prove this now.

Fix G and a dimension function δ for G . For ideals \mathfrak{a} and \mathfrak{b} of a ring R define the following group

$$S_\delta^{(k)}G(R, \mathfrak{a}, \mathfrak{b}) = \{h \in G(R, \mathfrak{a}\mathfrak{b}) \mid \varphi(h) \in \text{EE}(R', \mathfrak{a}', \mathfrak{b}') \text{ for all morphisms } \varphi : (R, \mathfrak{a}, \mathfrak{b}) \rightarrow (R', \mathfrak{a}', \mathfrak{b}') \text{ such that } \delta(R') \leq k\}.$$

Let $d = \delta(R)$. Clearly, $S_\delta^{(d)}G(R, \mathfrak{a}, \mathfrak{b}) = \text{EE}(R, \mathfrak{a}, \mathfrak{b})$. A crucial role in the the induction step of the proof of Theorem 12.5 plays the group $S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b})$. Here we enlarge this group in the same style as we enlarged the relative elementary group.

An element r of a ring R is called δ -regular if $\delta(R/r^k R) < \delta(R)$ for all positive integers k . Define a functor \tilde{E}_δ on the category of ideals by the formula

$$\tilde{E}_\delta(R, \mathfrak{a}) = \bigcap_r \left(G(R, r\mathfrak{a}) \tilde{E}(R, \mathfrak{a}) \right)$$

where the intersection is taken over all δ -regular elements r .

Lemma 12.2. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R and $d = \delta(R)$. Then $S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b}) \leq \tilde{E}_\delta(R, \mathfrak{a}\mathfrak{b})$.*

Proof. Let $r \in R$ be a δ -regular element. Put $\bar{r} = \rho_{\mathfrak{a}\mathfrak{b}}(r)$. Let $\mathfrak{q} = \{q \in R/\mathfrak{a}\mathfrak{b} \mid q\bar{r}^i = 0 \text{ for some } i \in \mathbb{N}\}$. It is easy to see that \mathfrak{q} is an ideal. Since $\delta(R) < \infty$, by 12.1(3) $R/\mathfrak{a}\mathfrak{b}$ is Noetherian, therefore \mathfrak{q} is finitely generated. Hence there exists $k \in \mathbb{N}$ such that $\bar{r}^k q = 0 \implies \bar{r}^{k-1} q = 0$ (this is the same as saying that $\bar{r}^k R/\mathfrak{a}\mathfrak{b}$ injects into $(R/\mathfrak{a}\mathfrak{b})_{\bar{r}}$, cf. [4, Lemma 4.10]). It follows that if $r^k p \in \mathfrak{a}\mathfrak{b}$, then $r^{k-1} p \in \mathfrak{a}\mathfrak{b}$ and $r^k p \in r\mathfrak{a}\mathfrak{b}$. Therefore, $\mathfrak{a}\mathfrak{b} \cap r^k R \subseteq r\mathfrak{a}\mathfrak{b}$.

Let $\rho = \rho_{r^k R}$ be the reduction homomorphism. Since $\delta(R/r^k R) \leq d - 1$, we have

$$\begin{aligned} \rho(S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b})) &\leq \text{EE}(R/r^k R, \rho(\mathfrak{a}), \rho(\mathfrak{b})) \text{ and} \\ S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b}) &\leq G(R, r^k R) \text{EE}(R, \mathfrak{a}, \mathfrak{b}). \end{aligned}$$

By definition, $S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b})$ and $\text{EE}(R, \mathfrak{a}, \mathfrak{b})$ are contained in $G(R, \mathfrak{a}\mathfrak{b})$, therefore we may replace $G(R, r^k R)$ by $G(R, r^k R \cap \mathfrak{a}\mathfrak{b})$ in the inclusion above. By the choice of k and by Lemma 11.2 we have

$$S_\delta^{(d-1)}G(R, \mathfrak{a}, \mathfrak{b}) \leq G(R, r\mathfrak{a}\mathfrak{b}) \tilde{E}(R, \mathfrak{a}\mathfrak{b}).$$

Since r was chosen arbitrarily, this inclusion implies the result. \square

The following result improves the commutator formula from Theorem 11.1.

Lemma 12.3. *Let \mathfrak{a} and \mathfrak{b} be ideals of a ring R . Suppose that G is simply connected. Then*

$$[\tilde{E}_\delta(R, \mathfrak{a}), G(R, \mathfrak{b})] \leq \text{EE}(R, \mathfrak{a}, \mathfrak{b}).$$

Proof. By condition 12.1(2) we have $G(S^{-1}R, S^{-1}\mathfrak{b}) = E(S^{-1}R, S^{-1}\mathfrak{b})$ for some multiplicative subset S of R . Since G and E commute with direct limits and $(S^{-1}R, S^{-1}\mathfrak{b})$ is a direct limit of principal localizations, given $b \in G(R, \mathfrak{b})$ there exists $r \in S$ such that $\lambda_r(b) \in E(R_s, \mathfrak{b}_s)$. By Lemma 10.4 with identity homomorphism φ there exists $m \in \mathbb{N}$ such that $[G(R, r^m\mathfrak{a}), b] \leq \text{EE}(R, \mathfrak{a}, \mathfrak{b})$. It follows from 12.1(2) that r^m is δ -regular. Therefore, $\tilde{E}_\delta(R, \mathfrak{a}) \leq G(R, r^m\mathfrak{a})\tilde{E}(R, \mathfrak{a})$. Now, the result follows from Theorem 11.1. \square

Corollary 12.4. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be ideals of a ring R . Then*

$$[S_\delta^{(k)}G(R, \mathfrak{a}, \mathfrak{b}), G(R, \mathfrak{c})] \leq S_\delta^{(k+1)}G(R, \mathfrak{a}\mathfrak{b}, \mathfrak{c}).$$

In particular, if $\delta(R) = 0 \implies G(R, \mathfrak{a}) = E(R, \mathfrak{a})$ for any pair (R, \mathfrak{a}) , then conditions 12.1(2) and (3) imply condition 12.1(1).

Proof. Let $\varphi : R \rightarrow R'$ be a ring homomorphism, $\delta(R') = k + 1$, $\mathfrak{a}' = \varphi(\mathfrak{a})R'$, $\mathfrak{b}' = \varphi(\mathfrak{b})R'$ and $\mathfrak{c}' = \varphi(\mathfrak{c})R'$. By Lemma 12.2 $S_\delta^{(k)}G(R', \mathfrak{a}', \mathfrak{b}') \leq \tilde{E}_\delta(R', \mathfrak{a}', \mathfrak{b}')$. By Lemma 12.3 we have

$$\begin{aligned} \varphi([S_\delta^{(k)}G(R, \mathfrak{a}, \mathfrak{b}), G(R, \mathfrak{c})]) &\leq [S_\delta^{(k)}G(R', \mathfrak{a}', \mathfrak{b}'), G(R', \mathfrak{c}')] \leq \\ &[\tilde{E}_\delta(R', \mathfrak{a}'\mathfrak{b}'), G(R', \mathfrak{c}')] \leq \text{EE}(R', \mathfrak{a}'\mathfrak{b}', \mathfrak{c}'). \end{aligned}$$

Now the required inclusion follows from the definition of $S_\delta^{(k+1)}G$.

If $G(R, \mathfrak{a}) = E(R, \mathfrak{a})$ provided $\delta(R) = 0$, then $S_\delta^{(0)}G(R, \mathfrak{a}, \mathfrak{b}) = G(R, \mathfrak{a}\mathfrak{b})$. We have already noticed that $S_\delta^{(1)}G(R, \mathfrak{a}, \mathfrak{b}) = \text{EE}(R, \mathfrak{a}, \mathfrak{b})$ if $\delta(R) = 1$. Note that we did not use condition 12.1(1) up to now, thus the inclusion in the statement follows from 12.1(2) and (3). Now, with $k = 0$ the inclusion turns to condition 12.1(1). \square

Theorem 12.5. *Let $\mathfrak{a}_0, \dots, \mathfrak{a}_d$ be ideals of a ring R , where $d \geq 1$. Let δ be a dimension function for G and $\delta(R) \leq d$. Then*

$$[G(R, \mathfrak{a}_0), G(R, \mathfrak{a}_1), \dots, G(R, \mathfrak{a}_d)] \leq \text{EE}(R, \mathfrak{a}_0 \dots \mathfrak{a}_{d-1}, \mathfrak{a}_d).$$

Proof. Proceed by induction on d . If $d = 1$ the claim coincides with condition 12.1(1). The induction step follows from Lemma 12.4. \square

The following straightforward corollary generalizes results of A. Bak, R. Hazrat, and N. Vavilov obtained in [4], [14] and [6].

Corollary 12.6. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_d$ be ideals of a ring R , where $d \geq 1$. Suppose one of the following holds.*

- *G is simply connected and Bass-Serre dimension of R is not greater than d .*
- *$G = \text{SL}_n$, $n \geq 3$, and Bass-Serre dimension of R is not greater than $d + n - 3$.*

Then $[G(R, \mathfrak{a}_0), G(R, \mathfrak{a}_1), \dots, G(R, \mathfrak{a}_d)] \leq \text{EE}(R, \mathfrak{a}_0 \dots \mathfrak{a}_{d-1}, \mathfrak{a}_d)$.

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ST.PETERSBURG STATE UNIVERSITY,

AND

ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES,

GC UNIVERSITY, LAHORE, PAKISTAN

E-mail address: `stepanov239@gmail.com`